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ON THE CONTACT INTERACTION BETWEEN A STRIP-SHAPED PUNCH AND A LINEARLY-DEFORMABLE BASE THROUGH A COVERING OF VARIABLE THICKNESS†

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The contact problem of the frictionless penetration of a punch with strip-shaped section into the surface of a linearly-deformable base protected by a thin elastic layer (covering) of variable thickness, the stiffness of which is comparable to or smaller than that of the supporting elastic body, is investigated. A Fredholm integral equation of the second kind is obtained for the unknown contact pressure with a coefficient in front of the leading term that is a fairly arbitrary function of the longitudinal coordinate. To solve it the Bubnov–Galerkin projection method is used in which the coordinate elements are chosen to be a system of orthogonal polynomials and delta-shaped functions [1, 2] (variational-difference method), together with an algorithm for the required asymptotic expansions [3] when the above-mentioned coefficient is small. In the special case of an elastic half-space protected by a covering of constant thickness, the results obtained are compared with the corresponding characteristics given in [4].

1. We first consider the three-dimensional problem of the action of a normal load $\sigma = e^{-i\beta y}\sigma(x)$ distributed along a strip-shaped region $|x| \leqslant a$, $|y| < \infty$ on the upper surface of a thin elastic layer of variable thickness $0 \leqslant z \leqslant h(x)$ $(0 < H_1 \leqslant h(x) \leqslant H_2, \lambda = H_2 a^{-1} \ll 1)$, rigidly attached to a base. The function $h(x)$ is taken to be at least continuous when $|x| < \infty$.

The problem reduces to integrating the Lamé equations when there are no body forces

$$(1-2\nu)\Delta\mathbf{u} + \operatorname{grad}\,\theta = 0,\ \left(\theta = \frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{1.1}$$

with boundary conditions

$$\begin{aligned} &z = 0: u = \upsilon = w = 0\ \ (|x| < \infty,\ |y| < \infty) \\ &z = h: \tau_{xz} = \tau_{yz} = 0\ \ (|x| < \infty,\ |y| < \infty) \\ &\sigma_z = e^{-i\beta y}\sigma(x)\ \ (|x| \leqslant a),\ \ \sigma_z = 0\ \ (|x| > a,\ |y| < \infty) \end{aligned} \tag{1.2}$$

(the stresses in the layer vanish at infinity), where

$$\sigma_z = 2G\left(\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\theta\right),\ \tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\ \tau_{yz} = G\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{1.3}$$

In formulae (1.1) and (1.3) $\mathbf{u} = \{u, \upsilon, w\}$ is the displacement vector of points in the elastic medium, while G and ν are its shear modulus and Poisson's ratio.

We shall look for the unknown functions occurring in (1.1)–(1.3) in the form

$$\{u, \upsilon, w\} = \{u_*, \upsilon_*, w_*\}e^{-i\beta y} \tag{1.4}$$

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Introducing the dimensionless variables and notation

$$x = ax_*,\ z = H_2 z_*,\ h_*(x_*) = \frac{h(x)}{H_2},\ \varepsilon = \frac{1-2\nu}{2(1-\nu)} \tag{1.5}$$

and then substituting (1.4) and (1.5) into the given relations (and dropping the asterisks), we obtain

$$\begin{aligned} &\lambda^2[u,_{xx} - \varepsilon(a\beta)^2 u] + \lambda(1-\varepsilon)(w,_z - i\lambda\beta a\upsilon),_x + \varepsilon u,_{zz} = 0 \\ &\lambda^2[\varepsilon\upsilon,_{xx} - (a\beta)^2\upsilon] - i\lambda\beta a(1-\varepsilon)(w,_z + \lambda u,_x) + \varepsilon\upsilon,_{zz} = 0 \\ &\varepsilon\lambda^2[w,_{xx} - (a\beta)^2 w] + \lambda(1-\varepsilon)(u,_x - i\beta a\upsilon),_z + w,_{zz} = 0 \end{aligned} \tag{1.6}$$

$$\begin{aligned} &z = 0:\ u = \upsilon = w = 0\ (|x| < \infty) \\ &z = h:\ (u,_z + \lambda w,_x) = (\upsilon,_z - i\lambda\beta aw) = 0\ (|x| < \infty) \\ &[w,_z + \lambda(1-2\varepsilon)(u,_x - i\beta a\upsilon)] = \begin{cases} \varepsilon H_2 G^{-1}\sigma & (|x| \leq 1) \\ 0 & (|x| > 1) \end{cases} \end{aligned} \tag{1.7}$$

Prior to the asymptotic analysis of expressions (1.6) and (1.7) we first note the inequalities [5]

$$c = a|\beta| < C < \infty,\ C\lambda \ll 1\ (C = \text{const}) \tag{1.8}$$

The second inequality is the "applicability condition for the theory of thin plates" and states that the external load is smoothly distributed over the surface of the covering $z = h$.

We represent the solution of system (1.6) satisfying conditions (1.7) in the form

$$u = \Phi(x,z) + O(\lambda),\ \upsilon = \Psi(x,z) + O(\lambda),\ w = \Gamma(x,z) + O(\lambda) \tag{1.9}$$

Substituting (1.9) into (1.6) and using (1.8), in the zeroth approximation we obtain

$$\Phi,_{zz} = 0,\ \Psi,_{zz} = 0,\ \Gamma,_{zz} = 0$$

from which we find

$$\Phi = A_0(x) + A_1(x)z,\ \Psi = B_0(x) + B_1(x)z,\ \Gamma = C_0(x) + C_1(x)z \tag{1.10}$$

Formulae (1.10), in agreement with (1.7)–(1.9), gives

$$\begin{aligned} &A_0(x) = A_1(x) = B_0(x) = B_1(x) = C_0(x) = 0 \\ &C_1(x) = \varepsilon H_2 G^{-1}\sigma(x) \end{aligned} \tag{1.11}$$

Substituting (1.11) into (1.9) and (1.10), putting $z = h(x)$ and returning to the former variables and notation (1.4) and (1.5) in the resulting expressions, we write

$$\begin{aligned} &w(x,y,h(x)) = \varepsilon G^{-1}h(x)\sigma(x)e^{-i\beta y} + O(\lambda) \\ &u = \upsilon = O(\lambda) \end{aligned} \tag{1.12}$$

Equations (1.12) show that a relatively thin elastic layer of variable thickness, attached to a rigid base can be modelled by the Fuss–Winkler support equations with foundation coefficient $\varepsilon G^{-1}h(x)e^{-i\beta y}$. Note that this result remains valid [6] in the case of a combined double-layer linearly-deformable base when

$$n = O(\lambda^m),\ \left(m > 0,\ n = \frac{\theta}{\theta_0},\ \theta = \frac{G}{1-\nu},\ \theta_0 = \frac{G_0}{1-\nu_0}\right)$$

Here G_0, ν_0 are the elastic characteristics of the foundation, whose settling under the action of the load

$$\sigma_z = e^{-i\beta y}\sigma(x) \quad (|x| \leq a, |y| < \infty)$$

has the form [7, 8]

$$w_0(x, y, 0) = \frac{e^{-i\beta y}}{\theta_0} \int_{-a}^{a} \sigma(\xi) k\left(\frac{\xi - x}{H}\right) d\xi \tag{1.13}$$

$$k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(u) e^{ist} ds, \quad u = \sqrt{s^2 + (\beta H)^2} \tag{1.14}$$

In relations (1.13) and (1.14) H is the characteristic parameter of the linearly-deformable base of the main elastic body and $K(u)$ is the symbol if its kernel.

Below we consider the case when $\Lambda = Ha^{-1} \gg \lambda$, while $K(s)$ is abounded function that is positive on the real axis and has the following asymptotic properties

$$K(s) \sim A \quad (s \to 0, \; A = \text{const})$$
$$K(s) = |s|^{-1} + O(|s|^{-\delta}) \quad (|s| \to \infty, \delta \geq 3) \tag{1.15}$$

We shall now study the contact problem of the frictionless penetration by the force $Pe^{-i\beta y}$, with eccentric application b, of a punch with strip-shaped cross-section, into the surface of a linearly-deformable base, protected by a thin "soft" elastic covering of variable thickness $h(x)$. We shall assume that the width of the contact domain is constant and equal to $2a$, and by virtue of the restriction $h(x)a^{-1} \leq \lambda \ll 1$ the contact condition

$$w_0 + w = g(x) e^{-i\beta y} \quad (|x| \leq a) \tag{1.16}$$

can be imposed on the surface of the linearly-deformable base $z = 0$.

Substituting the expressions for the vertical displacements (1.12) and (1.13) into relation (1.16) and introducing the dimensionless quantities

$$x' = xa^{-1}, \; \xi' = \xi a^{-1}, \; q(x') = \sigma(x)\theta_0^{-1}, \; f(x') = g(x)a^{-1}$$
$$\mu(x') = \varepsilon[an(1-\nu)]^{-1} h(x), \; P' = P(a\theta_0)^{-1}, \; b' = ba^{-1}$$

we arrive at an integral equation for the amplitude of the contact pressure $q(x)$ ($\mathbf{I}$ is the unit operator)

$$(\mu(x)\mathbf{I} + \mathbf{F})q = f(x) \quad (|x| \leq 1) \tag{1.17}$$

$$\mathbf{F}q = \int_{-1}^{1} q(\xi) k\left(\frac{\xi - x}{\Lambda}\right) d\xi \tag{1.18}$$

In formulae (1.17) and (1.18) and below the prime is omitted.

The statement of the problem must be completed by the formulation of the equilibrium conditions

$$P = \int_{-1}^{1} q(x) dx, \quad Pb = \int_{-1}^{1} xq(x) dx \tag{1.19}$$

which serve to find the rigid displacement of the punch $f(0)$ and its angle of rotation $f'(0)$.

2. Before solving integral equation (1.17), (1.18) we present a series of arguments establishing its correct solvability in the space of square-integrable functions $L_2(-1, 1)$. Note that by virtue of the properties of the kernel symbol $K(s)$, formulae (1.15), and also the positivity and boundedness of the function $\mu(x)$, we have the two-sided estimate [9]

$$m\|q(x)\|_{L_2}^2 \leq (\mathbf{R}q,q) \leq M\|q(x)\|_{L_2}^2 \tag{2.1}$$
$$\mathbf{R}q = (\mu(x)\mathbf{I}+\mathbf{F})q, \quad \text{const} = m > 0, \quad \text{const} = M < \infty$$

From inequality (2.1) the operator $\mathbf{R}$ is bounded and positive-definite in $L_2(-1, 1)$, which implies the existence of a bounded inverse operator $\mathbf{R}^{-1}$, i.e. the unique solvability of the original equation in $L_2(-1, 1)$. Here the correctness relation

$$\|q(x)\|_{L_2} \leq m^{-1}\|f(x)\|_{L_2} \tag{2.2}$$

holds.

Moreover, one can show that if $f(x) \in C(-1, 1)$, then also $q(x) \in C(-1, 1)$, where $C(-1, 1)$ is the space of functions continuous in $[-1, 1]$.

We represent the approximate solution of integral equation (1.17), (1.18) in the form

$$q_N(x) = \sum_{j=0}^{N} a_j \varphi_j(x)$$
$$\varphi_0(x) = \begin{cases} 2[1-(x+1)h^{-1}] & (x \in (-1,-1+h)) \\ 0 & (x \notin (-1,-1+h)) \end{cases} \tag{2.3}$$
$$\varphi_N(x) = \begin{cases} 2[1+(x-1)h^{-1}] & (x \in (1-h,1)) \\ 0 & (x \notin (1-h,1)) \end{cases}$$
$$\varphi_n(x) = \begin{cases} 1-(x-x_n)h^{-1} & (x \in (x_n-h,x_n+h)) \\ 0 & (x \notin (x_n-h,x_n+h)) \end{cases}$$

Here $\{\varphi_j(x)\}$ is a system of coordinate delta-shaped functions that is extremely dense in $L_2(-1, 1)$, and the nodes $x_j (j = 0 - N)$ cover the interval $[-1, 1]$ with step $h = 2N^{-1}$, and $x_0 = -1, x_N = 1$.

The coefficients a_j in expansion (2.3) are found from the condition that the discrepancy

$$\psi_N(x) = \mathbf{R}q_N - f(x)$$

is orthogonal to the first $N + 1$ terms of the sequence $\{\varphi_i(x)\}$

$$(\psi_N, \varphi_i)_{L_2} = 0 \quad (i = 0,1,2, \ldots, N) \tag{2.4}$$

As a result we arrive at an $(N + 1)$-order system of linear algebraic equations

$$\sum_{j=0}^{N} (\mathbf{R}\varphi_j, \varphi_i)_{L_2} a_j = (f, \varphi_i)_{L_2} \quad (i = 0,1,2, \ldots, N) \tag{2.5}$$

On the basis of inequalities (2.1) and (2.2) we conclude [2, 10] that a number N_* exists beyond which (i.e. when $N > N_*$) an approximation $q_N(x)$ of the form (2.3) is uniquely defined by relations (2.4). Moreover, we have the estimate

$$\|q - q_N\|_{L_2} \leq \|\mathbf{R}^{-1}\| \, \|\psi_N\|_{L_2} \to 0, \quad (N \to \infty)$$

which ensures the convergence of $q_N(x)$ to the exact solution $q(x)$ in the metric $L_2(-1, 1)$, from which it follows that $a_j \to q(x_j)$ at the internal nodes and $2a_0$, $2a_N \to q(x_0)$, $q(x_N)$. All this implies the applicability of the variational-difference method to investigate integral equations of the type (1.17), (1.18).

A key point in the algorithm being used is the estimate of the coefficients of system (2.5) expressed as triple integrals. However, application of the splines (2.3) to the original problem allows one to reduce the computational effort dramatically by reducing the quadratures to single integrals. Thus, starting with the definite integrals

$$\int_{-1}^{1}(1-|\tau|)e^{i\alpha\tau}d\tau = \frac{2(1-\cos\alpha)}{\alpha^2} = 2C(\alpha)$$

$$\int_{0}^{1}(1-\tau)e^{i\alpha\tau}d\tau = \frac{1-\cos\alpha}{\alpha^2}+i\frac{\alpha-\sin\alpha}{\alpha^2} = C(\alpha)+iS(\alpha)$$

we write

$$(f,\varphi_i)_{L_2} = h\int_{-1}^{1}f(x_i+hx)(1-|x|)dx \approx hf(x_i)$$

$$(\mathbf{R}\varphi_j,\varphi_i)_{L_2} = c_{ij}+e_{ij},\ c_{ij}=0\ (|i-j|>1)$$

$$c_{00}, c_{NN} = 4h\int_{0}^{1}\mu[\mp(1-hx)](1-x)^2dx$$

$$c_{01}, c_{N-1,N} = 2h\int_{0}^{1}\mu[\mp(1-hx)](x-x^2)dx,\ c_{01}=c_{10},\ c_{N-1,N}=c_{N,N-1}$$

$$c_{j,j\pm1} = h\int_{0}^{1}\mu(x_j\pm hx)(x-x^2)dx\ (j-1\neq 0) \tag{2.6}$$

$$c_{jj} = h\int_{-1}^{1}\mu(x_j+hx)(1-|x|)^2dx\ (j=1,2,\ \ldots,\ N-1)$$

$$e_{ij} = B\int_{0}^{\infty}K(\upsilon)C^2(s)\ \cos\frac{s(x_i-x_j)}{h}ds,\ \upsilon=\frac{\Lambda}{h}\sqrt{s^2+\gamma^2}$$

$$e_{0j} = B\int_{0}^{\infty}K(\upsilon)C(s)\left[C(s)\ \cos\frac{s(1+x_j)}{h}+S(s)\ \sin\frac{s(1+x_j)}{h}\right]ds$$

$$e_{ij}=e_{ji},\ e_{0j}=e_{j0},\ B=4h\Lambda\pi^{-1},\ \gamma=\beta Hh\Lambda^{-1}$$
$$i=1,2,\ \ldots,N-1;\ j=1,2,\ \ldots,N-1$$

$$e_{00}=e_{NN}=B\int_{0}^{\infty}K(\upsilon)[C^2(s)+S^2(s)]ds$$

$$e_{0N}=e_{N0}=B\int_{0}^{\infty}K(\upsilon)\left\{[C^2(s)+S^2(s)]\ \cos\frac{2s}{h}+2C(s)S(s)\ \sin\frac{2s}{h}\right\}ds$$

(where in the first formulae we have used the delta-shape of the sequence of coordinate functions).

We note that a somewhat different modification of the variational-difference method has been previously proposed [2], according to which the coordinate functions in formula (2.3) have the form

$$\varphi_j(x)=\varphi\left(\frac{x-x_j}{h}\right),\ x_j=-1+\frac{h}{2}(1+2j),\ h=\frac{2}{N+1}$$

$$\varphi(x)=\begin{cases}1-|x| & (|x|<1)\\ 0 & (|x|>1)\end{cases}$$

and the nodes x_j are internal points of the segment [−1, 1]. This choice of element system $\{\varphi_j(x)\}$ and the nodes x_j gives us the possibility of representing the coefficients c_{ij} and e_{ij} in the form of the sixth, seventh and eighth formulae of (2.6) for all i and j with values running from 0 to N. The calculations of the mechanical characteristics of the problem of Section 4 show that these two modifications of the variational-difference method agree with one another to a practical level of accuracy, so that below we shall confine ourselves to the first version.

In many cases [7, 9] the basis functions $\varphi_j(x)$ are taken to be a system of orthogonal polynomials (the orthogonal polynomial method). Here, for example, it is desirable to put

$$\varphi_j(x) = P_j^*(x), \quad P_j^*(x) = \sqrt{j+1/2}\,P_j(x)$$

where $P_j(x)$ are Legendre polynomials. We obtain for the coefficients of system (2.5) the expressions

$$(f,\varphi_i)_{L_2} = \int_{-1}^{1} f(x)P_i^*(x)dx \tag{2.7}$$

$$(\mathbf{R}\varphi_j,\varphi_i)_{L_2} = c_{ij}+e_{ij}, \quad c_{ij} = \int_{-1}^{1}\mu(x)P_i^*(x)P_j^*(x)dx$$

$$e_{ij} = (-1)^{[i/2]+[j/2]}\Lambda\sqrt{(2i+1)(2j+1)}\int_0^\infty K(u)J_{i+1/2}\left(\frac{s}{\Lambda}\right)J_{j+1/2}\left(\frac{s}{\Lambda}\right)\frac{ds}{s}$$

(where $[\alpha]$ is the integer part of α).

When estimating the coefficients e_{ij} of the form (2.6), (2.7) there is the problem of computing improper integrals of rapidly oscillating functions. An algorithm in [11] helps one to overcome this difficulty: it amounts to constructing a quadrature formula with an exact treatment of the oscillating component.

After finding the a_j ($j = 0, 1, \ldots, N$), from system (2.5) and constructing the exact solution of Eq. (1.17), (1.18) according to (2.3) one can determine from formulae (1.19) the integral characteristics of this solution. Thus, in the first case we obtain

$$P = h\sum_{j=0}^{N} a_j, \quad Pb = h\sum_{j=0}^{N} x_j a_j \tag{2.8}$$

and in the second

$$P = \sqrt{2}a_0, \quad Pb = \sqrt{2/3}\,a_1 \tag{2.9}$$

3. The method of Section 2 for solving the integral equation (1.17), (1.18) is efficient for sufficiently large values of the parameter $\mu(x)$. When $\mu(x) \to 0$ ($|x| \leq 1$), as is shown by numerical analysis of specific problems, the matrix of system (2.5) becomes ill conditioned, and the calculation becomes unstable. In this connection we shall give an algorithm for constructing the solution of the given equation that is effective for small values of the parameter $\mu(x)$. We will confine ourselves to finding the leading (zeroth) term of the asymptotic expansion of this solution.

Putting $\mu(x) = 0$ in (1.17) we can write the degenerate integral equation

$$\mathbf{F}q_0 = f(x) \quad (|x| \leq 1) \tag{3.1}$$

It was proved in [9] that if $f(c) \in H_1^\alpha(-1, 1)$($1/2 < \alpha, \leq 1$), the integral equation (3.1) is uniquely solvable in $L_p(-1, 1)$ ($1 < p < 2$) when $\Lambda \in (0, \infty)$ and its solution has the following structure

$$q_0(x) = \frac{\omega(x)}{\sqrt{1-x^2}}, \quad \omega(x) \in H_0^\gamma(-1,1) \quad \left(\frac{1}{2} < \gamma < 1\right) \tag{3.2}$$

where H_m^α (−1, 1) is the space of functions whose mth derivative satisfies the Hölder condition with index α for $|x| \leq 1$.

To solve Eq. (3.1) for any values of the parameter $\Lambda \in (0, \infty)$ one can, for example, use the orthogonal function method [9]. In accordance with the asymptotic formulae (1.15) we can therefore represent its kernel in the form

$$k(t) = \pi^{-1}K_0(tA^{-1}) + l(t) \tag{3.3}$$

$$l(t) \sim e^{-\kappa|t|}(|t| \to \infty,\ \kappa > 0),\ l(t) \in H_1^{\alpha}(-\infty, \infty)\ (\alpha = 1 - \varepsilon)$$

Here $K_0(tA^{-1})$ is the Macdonald function, reflecting all the main properties of the kernel $k(t)$, and the supplement $l(t)$ plays a secondary part.

Finding the regular part of the solution $\omega(x)$ in (3.2) in the form of a Fourier series in periodic Mathieu functions $\mathrm{ce}_i(\arccos x, -r)$ ($r = (2A\Lambda)^{-2}$ in the interval [−1, 1], and exactly inverting the principal part of the kernel of integral equation (3.1), (3.3) using the spectral relation

$$-\int_{-1}^{1} \frac{\mathrm{ce}_i(\arccos \xi, -r)}{\sqrt{1-\xi^2}} K_0\left(\frac{\xi - x}{A\Lambda}\right) d\xi = \frac{\pi \mathrm{Fek}_i(0,-r)}{\mathrm{Fek}_i'(0,-r)} \mathrm{ce}_i(\arccos x, -r)$$

we arrive at an infinite system of linear algebraic equations in the unknown coefficients of this expansion. To solve it we apply and justify the method of reductions.

We now consider the integral equation

$$(\mu(x)\mathbf{I} + \mathbf{F})q - \mu(\pm 1)q(\pm 1) - \mathbf{F}^{\pm}q = f(x) - f(\pm 1)\ (|x| \leqslant 1) \tag{3.4}$$

$$\mathbf{F}^{\pm}q = \int_{-1}^{1} q(\xi) k\left(\frac{\xi \mp 1}{\Lambda}\right) d\xi$$

in place of (1.17).

By an outer region we mean the interval

$$-1 + m_1\mu(-1) \leqslant x \leqslant 1 - m_2\mu(1)$$

in which we can take the degenerate solution (3.2) to be a sufficiently exact solution of Eq. (3.4). Internal regions are small neighbourhoods of the points $x = \pm 1$ with dimensions $m_1\mu(-1)$, $m_2\mu(1)$ ($m_1, m_2 \sim 1$); in these regions the effect of the covering on the distribution of contact stresses under the punch is comparable with the effect of the deformability of the elastic base. In the inner regions it is necessary to construct solutions of boundary-layer type "matched" at the boundary regions $x = \mp 1 \pm m_j\mu(\mp 1)$ ($j = 1, 2$) to the degenerate solution $q_0(x)$.

We confine ourselves to finding the boundary-layer solution only in the neighbourhood of the right-hand end of the punch $x = 1$, noting that the situation at $x = -1$ is similar. We transform (3.4) using (3.1) as follows:

$$\mu(x)q(x) + (\mathbf{F} - \mathbf{F}^{+})(q - q_0) = \mu(1)q(1)(|x| \leqslant 1) \tag{3.5}$$

Bearing in mind that when $m\mu \ll 1$ ($\mu \equiv \mu(1), m_2 \equiv m$)

$$q_0(1 - m\mu) = \omega(1)/\sqrt{2m\mu},\ \mu(1 - m\mu) \sim \mu$$

and that the function $q(x)$ at the boundary $x = 1 - m\mu$ between the inner and outer region is matched to $q_0(x)$, we look for a boundary-layer type solution in a neighbourhood of the point $x = 1$ in the form

$$q(x) = \frac{1}{\sqrt{\mu(x)}}\psi(t) + o\left[\frac{1}{\sqrt{\mu(x)}}\right],\ t = \frac{1-x}{\mu(x)} \tag{3.6}$$

Then the matching conditions acquire the form

$$q(1-m\mu)=\frac{\chi(m)\psi(0)}{\sqrt{\mu}}\sim q_0(1-m\mu)=\frac{\omega(1)}{\sqrt{2m\mu}}$$

from which it follows that

$$\chi(0)\sim 1,\ \chi(m)\sim 1/\sqrt{m}\ (m\gg 1),\ \psi(0)=\omega(1)/\sqrt{2} \tag{3.7}$$

Substituting (3.6) into (3.5), going over to new variables t, $\tau=(1-\zeta)\mu^{-1}(\xi)$, and letting μ tend to zero for fixed t and $\mu m\ll 1$, we obtain, in accordance with representation (3.3) and the definite integral [12]

$$\int_0^\infty \frac{1}{\sqrt{\tau}}\ln\left|\frac{\tau-t}{\tau}\right|d\tau=0$$

an integral equation governing the function $\chi(t)=\psi(t)[\psi(0)]^{-1}$

$$\chi(t)-\frac{1}{\pi}\int_0^\infty \chi(\tau)\ln\left|\frac{\tau-t}{\tau}\right|d\tau=1\ (0\leqslant t<\infty) \tag{3.8}$$

Note that the same Eq. (3.8) is also used to find the boundary layer at the other ($x=-1$) end of the punch.

The exact solution to integral equation (3.8) has the form [9, 13]

$$\chi(t)=\frac{1}{2\pi i}\int_L X(p)t^{-p}dp,\ X(p)=p\Gamma^2(p)S(p)$$

$$S(p)=\prod_{n=1}^{\infty}\left(1-\frac{1}{2n}\right)^{2p+1}\frac{\Gamma(-n-p+\frac{1}{2})\,\Gamma(-n+p+1)}{\Gamma(-n-p)\Gamma(-n+p+\frac{3}{2})}$$

where the contour L is the line $\operatorname{Re} p=\kappa(0<\kappa<\frac{1}{2})$.

After satisfying integral equations (3.1) and (3.8) the leading term of the uniformly fitting solution of Eq. (1.17), (1.18) for small values of the parameter $\mu(x)$ ($|x|\leqslant 1$), by virtue of relations (3.7) can be written in the form

$$q(x)=\frac{1}{\sqrt{1-x^2}}\left[\omega(x)-\frac{\omega(-1)}{\sqrt{2}}\sqrt{1-x}-\frac{\omega(1)}{\sqrt{2}}\sqrt{1+x}\right]+$$

$$+\frac{1}{\sqrt{2\mu(x)}}\left\{\omega(-1)\chi\left[\frac{1+x}{\mu(x)}\right]+\omega(1)\chi\left[\frac{1-x}{\mu(x)}\right]\right\} \tag{3.9}$$

Finally, knowing the function $q(x)$ of the form (3.9), one can calculate the force P and torque Pb applied to the punch using formulae (1.19).

4. We will present numerical calculations of the main mechanical properties of the contact problem posed in Section 1. The linearly-deformable base is an elastic half-space and the thickness of its protective covering has the form

$$\mu(x)=\mu_0(1-\eta\ \cos\alpha x)\ (|\eta|<1)$$

Here the representation $K(s)=s^{-1}$ holds for the symbol of the kernel of the original integral equation (1.17), (1.18), and using it according to formulae (1.14) when $\Lambda\to\infty$ we obtain

$$\mu_0(1-\eta\ \cos\alpha x)q(x)+\frac{1}{\pi}\int_{-1}^{1}q(\xi)K_0[c(\xi-x)]d\xi=f(x)\ (|x|\leqslant 1) \tag{4.1}$$

Table 1

μ_0	$x = 0.0$	0.2	0.4	0.6	0.8	1	Pf^{-1}
	0.359	0.360	0.364	0.371	0.385	0.420	0.745
2	0.356	0.358	0.361	0.368	0.381	0.411	0.740
	0.623	0.270	0.257	0.645	0.340	0.298	0.789
	0.631	0.280	0.264	0.657	0.345	0.308	0.797
	0.556	0.560	0.570	0.590	0.629	0.739	1.19
1	0.550	0.553	0.563	0.582	0.620	0.724	1.18
	0.920	0.421	0.403	0.980	0.546	0.525	1.23
	0.931	0.431	0.412	0.995	0.558	0.535	1.23
	0.761	0.767	0.789	0.833	0.927	1.224	1.70
0.5	0.748	0.755	0.776	0.819	0.907	1.196	1.67
	1.185	0.587	0.568	1.301	0.811	0.877	1.71
	1.192	0.595	0.576	1.315	0.822	0.885	1.71
	1.038	1.050	1.092	1.195	1.485	3.090	2.63
0.1	1.015	1.026	1.069	1.172	1.441	3.036	2.55
	0.957	0.968	0.989	1.066	1.294	1.768	2.19
0.01	1.048	1.052	1.067	1.156	1.351	4.023	2.56

We put $f(x) \equiv f = \text{const}$, $c = 1$, $\alpha = 10$ in (1.4). Table 1 records the values of the contact pressure $q(x)f^{-1}$ and force Pf^{-1} applied with eccentricity $b = 0$, estimated from formulae (2.3), (2.8) and (2.9) for various values of the dimensionless parameters μ_0 and η. The results of the first and second rows were, respectively, obtained for $\eta = 0$ by the variational-difference method with step $h = 0.1$ and by the orthogonal polynomial algorithm. Similar values for $\eta = 0.5$ and $\mu_0 = 2.1, 0.5$ are given in the third and fourth rows. It is clear that the given characteristics differ from one another by no more than 3% and differ from the corresponding quantities estimated in [4] at $\eta = 0$ by no more than 5%.

Note that when using the variational-difference method to solve integral equation (4.1) there is no need to calculate all the e_{ij} coefficients from formulae (2.6). It is sufficient just to estimate the diagonal elements $e_{ii}(i = 0 - N)$, and then to find the rest using the delta-form of the coordinate functions (2.3). We have

$$e_{ij} = h^2\pi^{-1}K_0[c(x_i - x_j)] \quad (i \neq j)$$

This form of the coefficients e_{ij} was used in [1], demonstrating the close interdependence of the algorithms developed in [1, 2], despite their apparent differences.

As was remarked in Section 3, to solve integral equation (4.1) for small values of $\mu(x)$ it is necessary to use the method of matched asymptotic expansions, which according to the above calculations works satisfactorily when $\mu(x) < 0.5$. The last two rows of Table 1 give the values of a uniformly fitting solution $q(x)f^{-1}$ obtained from (3.9) when $\eta = 0$ and $\mu_0 = 0.1, 0.01$, and also the relative strength of the embedding force Pf^{-1}, estimated according to the first condition in (1.19).

In summary we conclude that the numerical-analytic methods for solving integral equation (1.17), (1.18) developed in this paper cover all of the possible range of variation of the dimensionless parameter $\mu(x)$ and can be used to investigate more complicated problems in the theory of elasticity and mathematical physics with mixed boundary conditions.

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